STABILITY AND DECAY OF A CYLINDRICAL FILM OF LIQUID IN A GASEOUS MEDIUM

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Close to the orifice the film of liquid flowing from the nozzle of a pressure jet atomizer is approximately cylindrical in shape. Usually it also decays close to the nozzle and for a preliminary theoretical study it is convenient to formulate the problem of the stability of a cylindrical film of liquid moving in a stationary gaseous medium.

1. We shall consider a cylindrical film of an ideal liquid surrounded by an ideal fluid medium (Fig. 1) with outside radius a and inside radius b. We shall assume that the liquid moves along the x axis with velocity V, while the medium outside and inside the film is stationary.

We introduce the system of cylindrical coordinates (r, φ, x), with the x axis directed along the axis of the film, and the r axis along its radius. We denote by

$$\Phi_k = \Phi_k(r, \varphi, x, t) \qquad (k=1, 2, 3) \qquad (1.1)$$

the velocity potentials of the liquid and the medium. The subscript k = 1 refers to the liquid in the film, the subscripts k = 2 and k = 3 to the medium outside and inside the film, respectively. The densities of the liquid forming the film and the medium are denoted by ρ_1 and ρ_2, respectively.

The velocity potential Φ_k must satisfy the Laplace equation

$$\frac{\partial^2\Phi_k}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi_k}{\partial r} + \frac{\partial^2\Phi_k}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2\Phi_k}{\partial\varphi^2} = 0. \qquad (1.2)$$

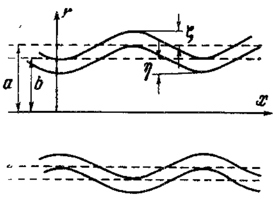

Fig. 1

The flow velocity components will have the form

$$V_{xk} = V_k + v_{xk}, \quad V_{rk} = v_{rk}, \quad V_{\varphi k} = v_{\varphi k},$$
$$V_1 = V, \quad V_2 = V_3 = 0, \qquad (1.3)$$
$$v_{\varphi k} = \frac{1}{r}\frac{\partial\Phi_k}{\partial\varphi}, \quad v_{xk} = \frac{\partial\Phi_k}{\partial x}, \quad v_{rk} = \frac{\partial\Phi_k}{\partial r} \quad (k=1, 2, 3).$$

The solution of Eq. (1.2) will be represented in the following form:

$$\Phi_k(r, \varphi, x, t) = f_k(r)\, e^{i(\alpha x + s\varphi - \beta t)}$$
$$(\alpha = 2\pi/\lambda, \ \beta = \beta_r + i\beta_i) \quad (k=1, 2, 3). \qquad (1.4)$$

Here α is the spatial circular frequency of the oscillations (wave number), λ is the wavelength of the

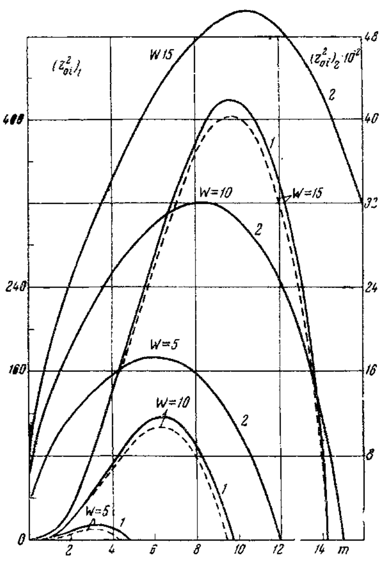

Fig. 2

superposed perturbation, and β is the complex frequency of the oscillations with respect to time. Substituting (1.4) in (1.2), we get

$$f_k'' + \frac{1}{r}f_k' - \left(\alpha^2 + \frac{s^2}{r^2}\right)f_k = 0 \qquad (k=1, 2, 3), \quad (1.5)$$

whose solution has the form

$$f_k = A_k I_s(\alpha r) + B_k K_s(\alpha r) \qquad (k=1, 2, 3), \qquad (1.6)$$

where A_k, B_k are arbitrary constants, and $I_s(x)$, $K_s(x)$ are Bessel functions of order s of imaginary argument.

Starting from the conditions of finite velocities at r = 0 and r = ∞, we write the velocity potentials for the motion of the liquid in the film and the surrounding medium as follows:

$$\Phi_1 = e^{i(\alpha x + s\varphi - \beta t)}\left[A_1 I_s(\alpha r) + B_1 K_s(\alpha r)\right],$$
$$\Phi_2 = e^{i(\alpha x + s\varphi - \beta t)} B_2 K_s(\alpha r), \qquad \Phi_3 = e^{i(\alpha x + s\varphi - \beta t)} A_3 I_s(\alpha r). \qquad (1.7)$$

2. At the boundary between the liquid film and the medium the following conditions must be satisfied.

At the outer and inner surfaces of the film the pressure difference must be balanced by the surface tension pressure

$$p_2 - p_1 = \sigma\left[-\frac{1}{a} + \frac{\zeta}{a^2} + \frac{\partial^2\zeta}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2\zeta}{\partial\varphi^2}\right] \quad \text{at} \quad r = a$$
$$p_3 - p_1 = \sigma\left[\frac{1}{b} - \frac{\eta}{b^2} - \frac{\partial^2\eta}{\partial x^2} - \frac{1}{b^2}\frac{\partial^2\eta}{\partial\varphi^2}\right] \quad \text{at} \quad r = b. \qquad (2.1)$$

Here p_1, p_2, p_3 are the pressures in the film and in the medium outside and inside the film, respectively, σ is the surface tension of the film liquid relative to the medium, ζ and η are the deviations of the liquid particles from the outer and inner surfaces of the undisturbed film (Fig. 1), respectively, and

$$r_1 = a + \zeta, \qquad r_2 = b + \eta. \tag{2.2}$$

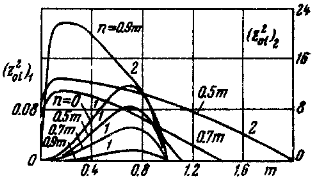

Fig. 3

The expressions for the pressures are obtained from the Lagrange-Cauchy integral in the form

$$\frac{p_1}{\rho_1} = -\left(\frac{\partial \Phi_1}{\partial t} + V\frac{\partial \Phi_1}{\partial x}\right) + \frac{p_0}{\rho_1}, \quad \frac{p_2}{\rho_2} = -\frac{\partial \Phi_2}{\partial t} + \frac{p_0}{\rho_2} - \frac{\sigma}{a\rho_2},$$
$$\frac{p_3}{\rho_2} = \frac{\partial \Phi_3}{\partial t} + \frac{p_0}{\rho_2} + \frac{\sigma}{b\rho_2}. \tag{2.3}$$

Here p_0 is the pressure in the undisturbed film.

We assume that the deviations ζ and η are periodic functions of t and x of the following type:

$$\zeta = \zeta^\circ e^{i(\alpha x + s\varphi - \beta t)}, \qquad \eta = \eta^\circ e^{i(\alpha x + s\varphi - \beta t)}, \tag{2.4}$$

where ζ° and η° are the amplitudes of the deviations of the liquid particles from the outer and inner surfaces of the undisturbed film, respectively.

Differentiating the expressions for the velocity potentials Φ_1, Φ_2, and Φ_3, from (1.7) we obtain, after substitution of the derivatives in (2.3), the expressions for the pressures

$$p_1 = i\rho_1 e^{i(\alpha x + s\varphi - \beta t)}(\beta - \alpha V)\,[A_1 I_s(\alpha r) + B_1 K_s(\alpha r)] + p_0,$$
$$p_2 = i\rho_2 \beta e^{i(\alpha x + s\varphi - \beta t)} B_2 K_s(\alpha r) + p_0 - \sigma / a, \tag{2.5}$$
$$p_3 = i\rho_2 \beta e^{i(\alpha x + s\varphi - \beta t)} A_3 I_s(\alpha r) + p_0 + \sigma / b.$$

Using (2.4) and (2.5), from (2.1) we obtain

$$i\rho_2\beta K_s(\alpha a)\,B_2 - i\rho_1(\beta - \alpha V)\,[A_1 I_s(\alpha a) + B_1 K_s(\alpha a)] + \sigma\left(\alpha^2 - \frac{1}{a^2} + \frac{s^2}{a^2}\right)\zeta^\circ = 0,$$
$$i\rho_2\beta I_s(\alpha b)\,A_3 - i\rho_1(\beta - \alpha V)\,[A_1 I_s(\alpha b) + B_1 K_s(\alpha b)] + \sigma\left(-\alpha^2 + \frac{1}{b^2} - \frac{s^2}{b^2}\right)\eta^\circ = 0. \tag{2.6}$$

The total differentials of the deviations of the liquid particles from the surfaces of the undisturbed film have the form

$$d\zeta = \frac{\partial \zeta}{\partial x}dx + \frac{\partial \zeta}{\partial t}dt, \qquad d\eta = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial t}dt.$$

Hence for the normal components of the velocity of displacement of the liquid particles at the outer and inner surfaces of the film we have

$$v_{r1} = \frac{\partial \zeta}{\partial x}V + \frac{\partial \zeta}{\partial t} \ \text{ at } r = a, \qquad v_{r1} = \frac{\partial \eta}{\partial x}V + \frac{\partial \eta}{\partial t} \ \text{ at } r = b,$$
$$v_{r2} = \frac{\partial \zeta}{\partial t} \ \text{ at } r = a, \qquad v_{r3} = \frac{\partial \eta}{\partial t} \ \text{ at } r = b.$$

Taking (2.4) into account, we get

$$v_{r1} = i\zeta^\circ e^{i(\alpha x + s\varphi - \beta t)}(\alpha V - \beta),$$
$$v_{r2} = -i\beta\zeta^\circ e^{i(\alpha x + s\varphi - \beta t)} \quad \text{at } r = a,$$
$$v_{r1} = i\eta^\circ e^{i(\alpha x + s\varphi - \beta t)}(\alpha V - \beta),$$
$$v_{r3} = -i\beta\eta^\circ e^{i(\alpha x + s\varphi - \beta t)} \quad \text{at } r = b. \tag{2.7}$$

On the other hand, using expressions (1.3) and (1.7), the expressions for the normal velocities can be written as follows:

$$v_{r1} = \alpha e^{i(\alpha x + s\varphi - \beta t)}\,[I_s'(\alpha r)\,A_1 + K_s'(\alpha r)\,B_1],$$
$$v_{r2} = \alpha e^{i(\alpha x + s\varphi - \beta t)} K_s'(\alpha r)\,B_2, \tag{2.8}$$
$$v_{r3} = \alpha e^{i(\alpha x + s\varphi - \beta t)} I_s'(\alpha r)\,A_3.$$

Here the prime denotes differentiation of the Bessel function with respect to the argument. Equating the right sides of (2.7) and (2.8), we obtain:

for the outer surface of the film

$$\alpha\,[I_s'(\alpha a)A_1 + K_s'(\alpha a)B_1] = i\zeta^\circ(\alpha V - \beta),$$
$$\alpha K_s'(\alpha a)B_2 = -i\beta\zeta^\circ; \tag{2.9}$$

for the inner surface of the film

$$\alpha\,[I_s'(\alpha b)A_1 + K_s'(\alpha b)B_1] = i\eta^\circ(\alpha V - \beta),$$
$$\alpha I_s'(\alpha b)A_3 = -i\beta\eta^\circ. \tag{2.10}$$

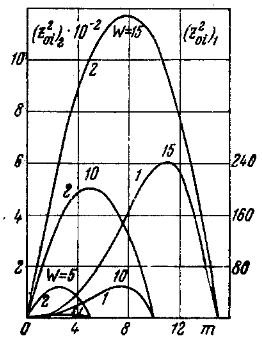

Fig. 4

From (2.9) and (2.10) we have

$$\zeta^\circ = \frac{\alpha\,[I_s'(\alpha a)\,A_1 + K_s'(\alpha a)\,B_1]}{i(\alpha V - \beta)} = -\frac{\alpha K_s'(\alpha a)\,B_2}{i\beta},$$
$$\eta^\circ = \frac{\alpha\,[I_s'(\alpha b)\,A_1 + K_s'(\alpha b)\,B_1]}{i(\alpha V - \beta)} = -\frac{\alpha I_s'(\alpha b)\,A_3}{i\beta}. \tag{2.11}$$

Substituting the expressions for ζ°, η° from (2.11) in (2.6), we obtain, together with Eqs. (2.11), a system of four linear (with respect to arbitrary constants) equations.

Introducing the dimensionless parameters

$$Z = \beta \sqrt{\rho_1 a^3 / \sigma}, \qquad W = \rho_2 a V^2 / \sigma, \qquad M = \rho_2 / \rho_1,$$
$$m = a\alpha, \qquad n = b\alpha, \qquad S = \sqrt{W / M}, \qquad (2.12)$$
$$\varepsilon = a / b = m / n,$$

and the notation

$$A_s(x) = \frac{I_s(x)}{K_s(x)}, \qquad A_{s+1}(x) = \frac{I_s'(x)}{-K_s'(x)},$$
$$\tau = Z - mS, \qquad B(x) = -\frac{K_s'(x)}{K_s(x)}, \qquad (2.13)$$

we obtain the system

$$\begin{aligned} &ZI_s'(n)A_1 + ZK_s'(n)B_1 + (mS - Z)I_s'(n)A_3 = 0, \\ &ZI_s'(m)A_1 + ZK_s'(m)B_1 + (mS - Z)K_s'(m)B_2 = 0, \\ &Z(mS - Z)I_s(m)A_1 + Z(mS - Z)K_s(m)B_1 + \\ &+ [MZ^2K_s(m) + m(m^2 - 1 + s^2)K_s'(m)]B_2 = 0, \\ &Z(mS - Z)I_s(n)A_1 + Z(mS - Z)K_s(n)B_1 + \\ &+ [MZ^2I_s(n) - m(m^2 - \varepsilon^2 + s^2\varepsilon^2)I_s'(n)]A_3 = 0. \end{aligned} \qquad (2.14)$$

Eliminating from the equations of the homogeneous system (2.14) the arbitrary constants, with notation (2.13) we get

$$\begin{vmatrix} A_{s+1}(n) & -1 & 0 & 1 \\ A_{s+1}(m) & -1 & -1 & 0 \\ \tau^2 A_s(m) & \tau^2 & a_{33} & 0 \\ \tau^2 A_s(n) & \tau^2 & 0 & a_{44} \end{vmatrix} = 0, \qquad (2.15)$$

$$a_{33} = M(mS + \tau)^2 - m(m^2 - 1 + s^2)B(m),$$
$$a_{44} = M(mS + \tau)^2 \frac{A_s(n)}{A_{s+1}(n)} - m(m^2 - \varepsilon^2 + s^2\varepsilon^2)B(n).$$

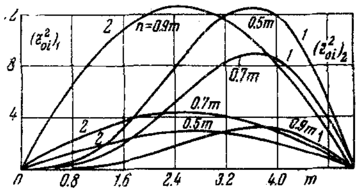

Fig. 5

It is easy to see that the roots of Eq. (2.15) can be represented in the form of a series in powers of $M^{1/2}$ (after the substitution $S = \sqrt{W / M}$):

$$\tau = Z_0 + Z_1 M^{1/2} + Z_2 M + \ldots \qquad (2.16)$$

In most cases the quantity M is very small (e.g., for a film of water in air $M = 1.2 \cdot 10^{-3}$), therefore in the expansion (2.16) we will confine ourselves to a single term, putting $\tau = Z_0$. Then from (2.15) we obtain

$$\begin{aligned} &[A_s(m) - A_s(n)]Z_0^4 + \{[A_s(n) + A_{s+1}(m)][m^2W + \\ &+ m(1 - m^2 - s^2)B(m)] + [A_s(m) + A_{s+1}(n)] \times \\ &\times [m^2WA_s(n) / A_{s+1}(n) + m(\varepsilon^2 - m^2 - s^2\varepsilon^2)B(n)]\}Z_0^2 + \\ &+ [A_{s+1}(m) - A_{s+1}(n)][m^2W + \\ &+ m(1 - m^2 - s^2)B(m)][m^2WA_s(n)/A_{s+1}(n) + \\ &+ m(\varepsilon^2 - m^2 - s^2\varepsilon^2)B(n)] = 0. \end{aligned} \qquad (2.17)$$

3. We shall consider special cases of the problem without account for tangential waves, i.e., the development of axisymmetric waves, for which in (2.17) we set s = 0, which gives

$$\begin{aligned} &[A_0(m) - A_0(n)]Z_0^4 + \{[A_0(n) + A_1(m)][m^2W + \\ &+ m(1 - m^2)B(m)] + [A_0(m) + A(n)][m^2WA_0(n)/A_1(n) + \\ &+ m(\varepsilon^2 - m^2)B(n)]\}Z_0^2 + [A_1(m) - A_1(n)] \times \\ &\times [m^2W + m(1 - m^2)B(m)][m^2WA_0(n)/A_1(n) + \\ &+ m(\varepsilon^2 - m^2)B(n)] = 0. \end{aligned} \qquad (3.1)$$

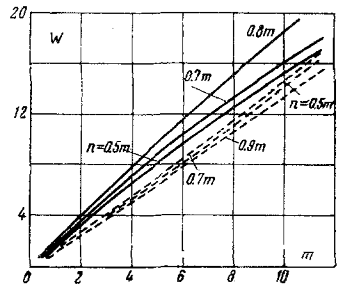

Fig. 6

We shall investigate a film with an internal cavity of small diameter, setting $\varepsilon = a/b \gg 1$ in Eq. (3.1), i.e., $a \gg b$ or $m \gg n$, and neglecting in its coefficients the quantities $A_0(n)$ and $A_1(n)$ as compared with the functions of m. We then obtain

$$\begin{aligned} &A_0(m)Z_0^4 + \{A_1(m)[m^2W + m(1 - m^2)B(m)] + \\ &+ A_0(m)[m^2WA_0(n)/A_1(n) + m(\varepsilon^2 - m^2)B(n)]\}Z_0^2 + \\ &+ A_1(m)[m^2W + m(1 - m^2)B(m)] \times \\ &\times [m^2WA_0(n) / A_1(n) + m(\varepsilon^2 - m^2)B(n)] = 0. \end{aligned} \qquad (3.2)$$

Solving Eq. (3.2) with respect to Z_0^2, we have

$$(Z_0^2)_1 = -A_1(m)/A_0(m)[m^2W + m(1 - m^2)B(m)], \qquad (3.3)$$
$$(Z_0^2)_2 = -[m^2WA_0(n)/A_1(n) + m(\varepsilon^2 - m^2)B(n)]. \qquad (3.4)$$

The root $(Z_0^2)_1$ gives the solution for the case of a solid cylindrical jet (b = 0, n = 0). This solution can also be obtained with the help of the corresponding passage to the limit from Eq. (3.1). Dividing this equation by the quantity

$$m^2W\frac{A_0(n)}{A_1(n)} + m(\varepsilon^2 - m^2)B(n)$$

and letting n tend to zero, we get the Eq. (3.3). Using the notation of (2.13), from (3.3) we get

$$Z_{0i}^2 = \frac{m^2WI_1(m)K_0(m) + m(1 - m^2)I_1(m)K_1(m)}{I_0(m)K_1(m)}, \qquad (3.5)$$

which corresponds to the equation obtained by Shekhtman [1]. Moreover, we have

$$Z_r = mS \ \text{ at } \ Z_0^2 < 0; \ Z_i = 0, \ Z_r = mS + Z_0 \ \text{ at } \ Z_0^2 > 0.$$

Figure 2 gives a graph of the square of the oscillation increment Z_{0i}^2 as a function of the dimensionless wave number m for different values of the Weber number W, calculated from (3.5) or (3.3). The broken lines on the same graph give the relation between $(Z_{0i}^2)_2$ and m calculated from (3.4).

We shall determine approximately the geometrical location of the maximum of the square of the oscillation increment as a function of the Weber number W at large values of the wave numbers m > 1.

We shall use the asymptotic formulas for Bessel functions [2]

$$I_0(x) \approx I_1(x) \approx e^x / \sqrt{2\pi x},$$
$$K_0(x) \approx K_1(x) \approx e^{-x}\sqrt{\pi / 2x}. \qquad (3.6)$$

From (2.13) at $s = 0$ we get

$$A_1(x)\,/\,A_1(x) = 1,\quad B(x) = 1$$

and instead of (3.3) for $m > 1$ we have

$$(Z_{0i}^2)_1 \approx m^2W + m(1 - m^2). \qquad (3.7)$$

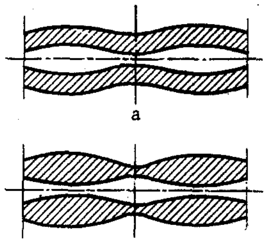

Fig. 7

Differentiating (3.7) and equating the derivative to zero, we get

$$2m_0W + 1 - 3m_0^2 = 0.$$

In the equation obtained it is possible to neglect unity as compared with the other two large terms, which gives

$$m_0 = {}^2/_3\, W, \qquad (3.8)$$

where m_0 denotes the critical value of the wave number corresponding to the maximum of the square of the increment.

Figure 2 gives a graph of the square of the oscillation increment $(Z_{0i}^2)_2$ as a function of the wave number m for different values of W as calculated from (3.4).

It is also possible to determine approximately the geometrical location of the maximum of the square of the increment $(Z_{0i}^2)_2$ for large values of the wave numbers. In this case,

$$(Z_{0i}^2)_2 = m^2W + m(\varepsilon^2 - m^2).$$

After equating to zero the derivative $2Wm_0 - 3m_0^2 + \varepsilon^2 = 0$, we have

$$m_0 = (W + \sqrt{W + 3\varepsilon^2})^{1/3}. \qquad (3.9)$$

As the next special case we shall consider the case of almost total absence of velocity. Setting $W = 0$ in (3.1), we have

$$\begin{aligned}[A_0(m) - A_0(n)]\,Z_0^4 + \{[A_0(n) + A_1(m)]\,m(1 - m^2)\,B(m) + \\ + [A_0(m) + A_1(n)]\,m(\varepsilon^2 - m^2)\,B(n)\}\,Z_0^2 + \qquad (3.10)\\ + [A_1(m) - A_1(n)]\,m^2(1 - m^2)(\varepsilon^2 - m^2)\,B(m)\,B(n) = 0.\end{aligned}$$

In Fig. 3 we present graphs of $(Z_{0i}^2)_1$ versus wave number for different film thicknesses, i.e., different values of the parameter ε at $W = 0$, as calculated from (3.10). The same figure includes the graph for the limiting case of a continuous solid jet (Rayleigh case, $n = 0$).

Figure 3 gives a graph of the analogous relation for the second root of Eq. (3.10).

From Eq. (3.10) it is possible to obtain the case of oscillations of a plane film by setting $m \gg 1$, $n \gg 1$, $m - n \approx 1$ (the radii of the film a and b are large as compared with the wavelength λ). Then Eq. (3.10) assumes the following form:

$$Z_0^4 - 2m^3Z_0^2 + m^6 = 0. \qquad (3.11)$$

Hence $Z_0^2 = m^3$, i.e., $Z_i = 0$, $Z_r = \pm m^{3/2}$, from which it follows that in the absence of velocity the plane film is stable [3].

We shall also consider the case of large flow velocities of the liquid film: $W \gg 1$, $m \gg 1$, $n \gg 1$, $m - n \approx 1$. Using the asymptotic expressions (3.6) for the Bessel functions at large arguments, we obtain

$$A_0(x) \approx A_1(x) \approx e^{2x}/\pi, \qquad B(x) = 1.$$

Substituting the expressions obtained in Eq. (3.1), we have

$$Z_0^4 + 2m^2(W - m)\,\mathrm{cth}\,(m - n)\,Z_0^2 + m^4(W - m)^2 = 0. \qquad (3.12)$$

Whence we obtain

$$\begin{aligned}(Z_{0i}^2)_1 &= m^2(W - m)\,\mathrm{th}^{1/2}(m - n),\\ (Z_{0i}^2)_2 &= m^2(W - m)\,\mathrm{cth}^{1/2}(m - n),\end{aligned} \qquad (3.13)$$

which gives instability at $W > m$ and stability at $W < m$.

Figure 4 gives the square of the increment as a function of wave number for different values of W and film thickness corresponding to $n = 0.9$ m ($\varepsilon = 1/0.9$) as calculated from (3.13). Figure 5 gives the same relations for $W = 5$ and different film thicknesses, $(Z_{0i}^2)_2$ being given to the scale 0.1.

In order to find the geometric locations of the maxima on the curves expressing the above-indicated relations, we differentiate the expressions for the squares of the increments (3.13) with respect to m and equate the derivatives to zero. Solving the equations obtained with respect to W (which is simpler, since they are transcendental with respect to m_0), we obtain

$$W = m_0\frac{km_0 + {}^3/_2\,\mathrm{sh}\,2km_0}{km_0 + \mathrm{sh}\,2km_0} \quad \text{for } (Z_{0i}^2)_1 \qquad (3.14)$$

$$W = m_0\frac{3\,\mathrm{sh}\,2km_0 - 2km_0}{2\,[\mathrm{sh}\,2km_0 - km_0]} \quad \text{for } (Z_{0i}^2)_2 \qquad (3.15)$$

$$\left(k = \frac{\varepsilon - 1}{2\varepsilon}\right).$$

With increase in wave number both expressions (3.14) and (3.15) tend to the same limit $W = 3m_0/2$, which is in agreement with (3.8).

Relations (3.14) and (3.15) are given in Fig. 6 for different values of ε.

Finally, we shall consider the case of oscillations of the liquid film with allowance for the effect of transverse waves, assuming high flow velocities ($W > 1$).

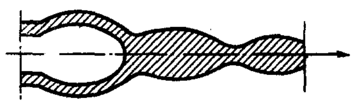

Fig. 8

In this case it is necessary to consider only the region of large wave numbers $m \gg 1$, $n \gg 1$, within which lies the curve representing the relation between increments and wave number at $W > 1$. Using Eq. (2.17) and the asymptotic expressions for Bessel functions with large argument, we obtain

$$A_s(x) \approx A_{s+1}(x) \approx e^{2x}/\pi, \qquad B(x) = 1$$

$$\begin{aligned}Z_0^4 + \mathrm{cth}\,(m - n)\,[2m^2W + m(1 - m^2 - s^2) + \\ + m(\varepsilon^2 - m^2 - s^2\varepsilon^2)]\,Z_0^2 + \qquad (3.16)\\ + [m^2W + m(1 - m^2 - s^2)]\,[m^2W + m(\varepsilon^2 - m^2 - s^2\varepsilon^2)] = 0.\end{aligned}$$

Assuming that $m - n \gg 1$ and $\mathrm{cth}\,(m - n) \approx 1$ and that it is possible to neglect unity as compared with s^2, from the previous equation we obtain

$$\begin{aligned}Z_0^4 + [2m^2(W - m) - ms^2(1 + \varepsilon^2)]\,Z_0^2 + \\ + [-m^2(W - m) + ms^2\varepsilon^2]\,[-m^2(W - m) + ms^2] = 0.\end{aligned} \qquad (3.17)$$

Solving Eq. (3.17), we obtain

$$\begin{aligned}(Z_0^2)_1 &= -m^2(W - m) + ms^2\varepsilon^2,\\ (Z_0^2)_2 &= -m^2(W - m) + ms^2.\end{aligned} \qquad (3.18)$$

The graphs of (3.18) are given by the broken line in Fig. 2.

4. We shall consider the perturbation modes predominating in cases for which the oscillation increments were determined above. The determinant (2.15) (at $s = 0$, $\tau = Z_0$) corresponds to the system of equations

$$\begin{gathered} C_1 A_1(n) - C_2 + C_4 = 0, \\ C_1 Z_0^2 A_0(m) + C_2 Z_0^2 + C_3 [m^2 W + m(1 - m^2) B(m)] = 0, \\ C_1 A_1(m) - C_2 - C_3 = 0, \\ C_1 Z_0^2 A_0(n) + C_2 Z_0^2 + \\ + C_4 [m^2 W A_0(n) / A_1(n) - m(m^2 - \varepsilon^2) B(n)] = 0. \end{gathered} \tag{4.1}$$

Taking the first three equations of system (4.1), dividing them by C_1, subtracting the first from the second and adding the second to the third, we get

$$\begin{gathered} A_1(m) - A_1(n) - \frac{C_3}{C_1} - \frac{C_4}{C_1} = 0, \\ A_0(m) + A_1(m) - \frac{C_3}{C_1} + \frac{C_3}{C_1}[m^2 W + m(1 - m^2) B(m)] \frac{1}{Z_0^2} = 0. \end{gathered}$$

Whence

$$\begin{gathered} \frac{C_3}{C_1} = \frac{Z_0^2 [A_0(m) + A_1(m)]}{Z_0^2 - m^2 W - m(1 - m^2) B(m)}, \\ \frac{C_4}{C_1} = A_1(m) - A_1(n) - \frac{Z_0^2 [A_0(m) + A_1(m)]}{Z_0^2 - m^2 W - m(1 - m^2) B(m)}. \end{gathered} \tag{4.2}$$

Using expressions (2.11) for the amplitudes of the deviations of the liquid particles at the outer and inner surfaces of the film for $s = 0$, we write their ratio

$$\frac{\zeta^\circ}{\eta^\circ} = -\frac{C_3}{C_4} \frac{K_1(m)}{I_1(n)}. \tag{4.3}$$

Having determined C_3/C_4 from (4.2) and substituting its value in (4.3), we obtain the following expression for the quantity ξ characterizing the sign of the ratio of the liquid particle deviations:

$$\begin{gathered} \xi = \frac{\zeta^\circ}{\eta^\circ} \frac{I_1(n)}{K_1(m)[A_0(m) + A_1(m)]} = \\ = \frac{Z_0^2}{Z_0^2 [A_0(m) + A_1(n)] + [A_1(m) - A_1(n)][m^2 W + m(1 - m^2) B(m)]}, \end{gathered} \tag{4.4}$$

since the quantity

$$I_1(n) / K_1(m) [A_0(m) + A_1(m)] > 0.$$

Now, substituting in (4.4) the square of the oscillation increment Z_0^2, obtained for some case, we can determine the sign of the ratio of the amplitudes of the liquid particle deviations from the outer and inner surfaces of the film and establish the oscillation phase shift at these surfaces.

For the case of a liquid film with an internal cavity of small diameter we substitute the first root (3.3) in (4.4) and neglect in the denominator the quantity $A_1(n)$ as small compared with the quantities $A_0(m)$ and $A_1(m)$. We then get $\xi = \infty$, which shows the smallness of the amplitudes of the waves (corresponding to the first root) propagating along the inner surface of the film as compared with the amplitude of the waves on the outer surface. In the limit the amplitudes of these waves tend to zero with decrease in the diameter of the internal cavity.

Substituting the second root (3.4) in (4.4) and also neglecting $A_1(n)$ in the denominator, we obtain

$$\begin{gathered} \xi = \\ = -[m^2 W A_0(n) / A_1(n) + m(\varepsilon^2 - m^2) B(n)] \times \\ \times \Big(-A_0(m)[m^2 W A_0(n) / A_1(n) + m(\varepsilon^2 - m^2) B(n)] + \\ + A_1(m)[m^2 W + m(1 - m^2) B(m)]\Big)^{-1}, \end{gathered} \tag{4.5}$$

The second term of the denominator in (4.5) is considerably smaller than the first; therefore $\xi > 0$ and the amplitudes ζ°, η° have the same sign. Hence, along the outer and inner surfaces of a film with an internal cavity of small diameter the waves (corresponding to the second root) are in phase (Fig. 7a).

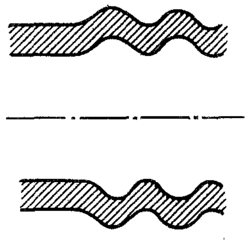

Fig. 9

For the case of an almost stationary film (analog of the Rayleigh problem), taking values of the roots $(Z_0^2)_{1,2}$ of Eq. (3.10) from the graphs of Fig. 3 and substituting for the corresponding values of the wave numbers m and n in (4.4), we easily see that the first root corresponds to the inequality $\xi < 0$ [the surfaces of the film oscillate out of phase (Fig. 7b)], while the second root corresponds to the inequality $\xi > 0$ [these surfaces oscillate in phase (Fig. 7a)].

For the case of motion of a film at large velocities the substitution of roots $(Z_0^2)_{1,2}$ from (3.13) in Eq. (4.4) also shows that the first root corresponds to the inequality $\xi < 0$, and the second to $\xi > 0$.

Hence, it may be said that for three special cases considered the first root (plus sign in front of radical in the solution of Eq. (3.1), which is quadratic in Z_0^2) corresponds to the inequality $\xi < 0$, which gives different signs of the amplitudes ζ°, η° at the outer and inner surfaces of the film, i.e., that these surfaces fluctuate in phase opposition (Fig. 7b). The second root [minus sign in front of radical in the solution of quadratic Eq. (3.1)] corresponds to the inequality $\xi < < 0$, which gives the same signs of the amplitudes ζ°, η° at the outer and inner surfaces of the film, i.e., these surfaces fluctuate in phase (Fig. 7a).

5. We shall also attempt to draw certain (mainly qualitative) conclusions concerning the mechanism of decay of the liquid film starting from the theoretical results obtained. It should be noted that in real conditions the initial section of the flow of a liquid film into a gaseous medium from some nozzle is unsteady along the length of the jet, where as our theoretical problem assumes the presence of steady flow, which is taken as the original undisturbed motion.

From a consideration of the data on the variation of the oscillation increment as a function of wave number for the special cases considered we may conclude the following.

At small values of the Weber number W (if the effect of transverse waves is disregarded $s = 0$) from the fact that the second root $(Z_{0i}^2)_2$ is considerably greater than the first $(Z_{0i}^2)_1$ (see Fig. 3) and Rayleigh's principle we may conclude that after leaving the nozzle the film of liquid develops waves that are in phase on the outer and inner surfaces. Since the amplitudes of the oscillations quickly increase (the increment is large), this leads to the collapse of the internal cavity (see Fig. 8), after which the film is transformed into a continuous solid jet decaying according to Rayleigh's law (at very small Weber numbers) or according to the Petrov-Shekhpman law. In this case the decay is described by the oscillation mode corresponding to the first root (3.3) of Eq. (3.2).

At large Weber numbers and at $s = 0$ it is necessary to consider two cases: for $3 < W_h < 10$ the oscillations of the surface of the film are in phase [second root of (3.13)] with wavelength of the order of its thickness ($\lambda \sim h = a - b$) and upon decay the integrity of the film may immediately be destroyed as a result of its being strongly pulled out (Fig. 9). For $W_h > 10$ on both surfaces of the film waves develop with a wavelength that is small compared with the film thickness. These waves may correspond to either of the two types considered (in phase, which corresponds to the second root, or out of phase, which corresponds to the first root); the occurrence of waves of either type is apparently equiprobable in view of the closeness of the values of the oscillation increments (here $W_h = \rho_2 h V^2 / \sigma$). In this case the decay mechanism evidently corresponds to that proposed by Taylor [4], consisting in the separation of liquid droplets with diameters of the order of the wavelength from both film surfaces without preliminary destruction of its integrity. The wavelength corresponds to the wave number $m_0 = 2W/3$, i.e., is the same as for the case of decay of a plane film. Here the cylindricity of the film ceases to affect the decay and liquid droplets of the diameter

$$d \sim \lambda = \frac{3\pi\sigma}{\rho_2 V^2} \qquad (5.1)$$

separate from both surfaces.

The separation from the film of rings of liquid is evidently improbable and may perhaps occur only in a narrow range of Weber numbers, somewhere near $W_h = 10$. As may be seen from Fig. 2, transverse waves have little effect on the oscillation increment, tending to reduce it.

It should be pointed out that the conclusion about the size of the droplets separating from the surfaces of the film is not completely accurate owing to the neglect of friction forces in the liquid. Experiments show that the size of the droplets is affected by the Laplace number $L = a\sigma\rho_1 / \mu_1^2$ (here a is the diameter of the jet, σ is the surface tension, ρ_1 is the density of the liquid, and μ_1 is the absolute viscosity of the liquid).

The action of the viscosity of the liquid on its decay may be regarded from two viewpoints. Firstly, the viscous forces lead to a change in the basic flow—a boundary layer, whose presence must lead to a change in wave formation, is formed.

Secondly, the viscous forces may have a direct effect on the development of perturbations for a given basic flow profile. In this case a study of the stability must be based not on the equations of an ideal liquid, but on the Navier-Stokes equations, which seriously complicates the investigation. For liquids that are not too viscous, this effect is evidently very slight. In view of this it seems to us that the main role is played only by the change in velocity profile and that the behavior of the perturbations is described by the same equations of an ideal liquid as have been used above.

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